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Quantum spinors and spin groups from quantum Clifford algebras

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Abstract. A general construction of multiparametric quantum spinors and corresponding quantum $\mathrm{Spin}_\mu(2\nu-h, h)$ groups associated to 2ν-(Pseudo)-Euclidean spaces is presented, and their homomorphism to the respective SO_μ groups is discussed. This construction is based on a quantum Clifford algebra and is described in detail for involutive (pure twists) intertwining braids. For general braid operators that admit abstract 'volume elements', a procedure is also given for deriving quantum analogues of these groups.

1. Introduction

In their most general mathematical form, spinors were invented by Cartan in 1913, when investigating linear representations fo simple groups. Cartan's geometrical approach to spinors [1], which is based on their connection with isotropic geometrical elements, has been of fundamental importance for the applications of spinors in the theory of relativity [2] and for the development of the theory of twistors as a Robinson congruence of null lines in Minkowski space [3]. Another impressive example of the usefulness of spinor analysis in a new domain has been provided by Witten in his proof of the positive energy theorem in Einstein's relativity [4]. Nevertheless, the geometrical point of view of Cartan, which stressed the equivalence of projective pure spinors of complex Euclidean spaces with null planes of maximal dimension in these spaces, and the corresponding projective geometry, remarkably rich and elegant, was primarily the subject of mathematical studies.

On the other hand, the modern Kaluza–Klein theories of unification of gauge fields and gravitation, the theories of grand unification and, more recently, superfield, super-string, membrane and conformal field theories have led to theories which make essential use of geometries of more than four dimensions. Such higher-dimensional theories are believed to provide a good chance for the solution of several of the most crucial questions and problems in four-dimensional unified field theories. This enhanced role of the geometry of multidimensional spaces—and hence also that of spinor structures and that of 'pure' spinors in higher dimensions—in fundamental theoretical physics; exploiting, in particular, the geometrical properties previously developed by Cartan and others [5], also see [6] and references contained therein.

Most discussions of Planck-scale physics work, however, within an underlying classical geometry. This may not be an altogether justified assumption. A kind of deeper quantum geometry [7], from which classical geometry should emerge, may be required, not only at the Planck scale and quantum cosmology level, but also to provide the correct language

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for resolving the paradoxes in quantum mechanics related to the macroscopic geometry of measuring apparati and quantum mechanical evolution. Non-commutative geometry based on quantum groups and braided groups has appeared in the last decade or so as a plausible mathematical formalism for formulating questions and making predictions about physics beyond the Planck scale, by providing the possibility of extending the concepts of gauge theory, curvature, non-Euclidean geometry and their description in terms of fibre bundles to the situation where coordinates are non-commuting operators.

Within this program it is expected that a quantum deformation of a Lorentzian manifold, where functions on spacetime are replaced by some non-commutative algebra, should have some properties similar to those of the ordinary classical space. Depending on the properties so chosen, several models for the four-dimensional quantum Minkowski plane and the corresponding quantum Lorentz [8] and quantum Poincaré [9–11] groups acting on them having been proposed in the literature.

Also, as soon as coordinate algebras are made non-commutative the choice of various possible differentiable structures appears as a new degree of freedom. These possible quantum differential calculi are determined by the selection of the intertwiners in the quantum algebras, and offer various natural scenarios for constructing fibre bundle formulations of gauge theories over quantum spaces where symmetry breaking could be achieved by quantum deforming the classical one.

The main objective of this paper is to consider the possibility of a combined need for higher-dimensional spaces ($d \geqslant 4$) and their quantum deformations for the description of physics at and below the Planck length, with quantum spinors and their associated quantum spin-symmetry groups playing an important role in the ensuing quantum geometry. Under such circumstances a theory for constructing quantum spinor algebras and the spin quantum groups related to (pseudo)-Euclidean spaces with 2ν-dimensions, with $\nu \in \mathbb{Z}^+$, and metrics with arbitrary signatures, becomes important. Although pure twisted and the quantum group Spin_μ (4) has been previously treated in the literature (cf the works cited in [8] and [9–11]), this is not so with higher-dimensional quantum spin groups. It is desirable, in addition, to have an axiomatic formulation of a deformed spinor theory and quantum spin groups which preserves Cartan's geometrical approach as much as possible and, at the same time, allows one to investigate the implications of different possible choices of intertwiners and differential calculi, as well as deformations of some of the axioms. For this purpose, and since spinors are intimately related to Clifford algebras, we have chosen to use as a starting point for our discussion the theory of quantum Clifford algebras which we have previously developed [12]. We have concentrated our attention here on involutive intertwiners (better known as pure twists) for two reasons: first, because calculations, which for higher-dimensional spaces become much more complicated, remain still tractable for involutive braids and serve to illustrate the main features of our formalism; second, the resulting *-compatible differential calculus for the coordinates—of the subjacent (pseudo)-Euclidean spaces—allows for the usual interpretation of differentials as shifts of coordinates, and left and right actions of derivations are, in this case, two representations of the same abstract operator. Thus we avoid the problems of interpretation associated with the nonlinear and rather cumbersome derivation operators that occur when considering, for example, Hecke braidings [11]. It should be clear, however, from the generality of our theory of quantum Clifford algebras that our constructions can be readily extended to more complicated types of intertwiners. We discuss such a procedure at the end of the paper.

Another feature of our analysis is that it serves to establish the important general commutation behaviour of the entries in the quantum block matrices for $\mathrm{Spin}_\mu(n)$. We show that the entries in each block commute with themselves only for the case $n = 4$, while semi-spinors of a given type commute among themselves only for $n = 4$, they commute with those of the other type for $n = 6$ and are no longer commutative in either case for $n \geqslant 8$.

Finally note that our formalism is also relevant to the study of the theory of deformed twistors. This follows from the fact that for $n = 6$, taking the relevant orthogonal group to be $SO(4, 2)$, the semi-spinors are the univalent four-dimensional twistors and dual twistors, while for $n = 8$ the relevant groups to twistor theory are $SO(8, C)$ and $SO(4, 4)$.

In order to make the paper as self-contained as possible, we have structured it as follows. In section 2 we construct a quantum Clifford algebra and a compatible *-structure. By requiring that the fundamental property of spinor transformations be preserved in the quantum case, a non-commutative algebra is induced for the 'coordinates' of the underlying pseudo-Euclidean spaces. Taking the generators of this algebra as a comodule, linear twisted group matrices and their orthogonal subgroups with a *-compatible structure are obtained.

As a separate part of section 2 we include a general construction of quantum analogues of linear (and orthogonal) groups, starting from appropriate braid operators admitting abstract 'volume elements'. Conceptually, we shall follow the expositions in [13]. The innovative part of this subsection is in the systematical use of bicovariant bimodules [14], so that all braidings in the formalism become the braidings intrinsically associated to the appropriate bicovariant bimodules. As we shall see, this technique allows us to derive all basic properties and relations involving quantum determinants in a simple and elegant way and help us to extend our formalism to more general braids.

In section 3 we develop a pure twisted deformation of Cartan's spinor algebra and the corresponding quantum $\mathrm{Spin}_\mu(n)$ groups (for n even), both with a compatible *-structure appropriate to a given signature of the pseudo-Euclidean space metric.

Section 4 is devoted to outline an approach to extend our formalism to the case of more general braids.

Finally, in the appendix the general theory for quantum $\mathrm{Spin}_\mu(n)$ groups, given in section 3, is applied to the specific cases of involutive twisted braids to quantum $\mathrm{Spin}_\mu(4 - h, h)$ $\mathrm{Spin}_\mu(6 - h, h)$, $\mathrm{Spin}_\mu(8 - h, h)$, and the morphism of these quantum spin group matrices to the respective quantum orthogonal matrices is explicitly given. We also provide the explicit relations which result between the deformation parameters of the involutive braid matrices associated with the Clifford algebras and those occurring in the spin group matrices†.

2. $GL_\mu(2\nu - h, h)$ groups

In [12] we presented a theory of quantum Clifford algebras, based on a quantum generalization of Cartan's theory of spinors. For even-dimensional spaces (the generalization to odd-dimensional spaces can be readily performed) the construction starts by considering the two isotropic subspaces V and V' ($\dim(V) = \dim(V') = \nu$) into which a 2ν-dimensional Euclidean or pseudo-Euclidean space W decomposes, i.e. $W = V \oplus V'$, where, because of the isotropy, V' can be envisaged as the dual to V. The subspaces V, V' and $\mathbb{C}$ with the usual tensor products generate a braided monoidal category, where the braiding $\sigma : V \otimes V \to V \otimes V$ satisfies the Hecke condition $\sigma^2 = (1 - \mu^2)\sigma + \mu^2 I$.

A standard representation of such an operator (coming from quantum $SU(n)$ groups [16]) is given by $\sigma(e_i \otimes e_j) = \mu(e_j \otimes e_i)$ for $i < j$, while $\sigma(e_i \otimes e_i) = e_i \otimes e_i$ and $\sigma(e_i \otimes e_j) = \mu(e_j \otimes e_i) + (1 - \mu^2)(e_i \times e_j)$ for $i > j$. Here $\{e_i\}_{i=1}^{\nu}$ are basis elements of V.

† Throughout the text we shall be using indiscriminatingly the terms involutive and pure twists [15] to mean the same type of braidings.

The generators $H(e_i)$ of the quantum Clifford algebra $Cl(V,\sigma)$ satisfy the relations

$$H(e_i)H(e_j) + \frac{1}{\mu^2}\sum_{k,l}(\sigma)_{ij}{}^{kl}H(e_k)H(e_l) = 0 \tag{1}$$

where $\sigma(e_i \otimes e_j) = \sum_{k,l}(\sigma)_{ij}{}^{kl}e_k \otimes e_l$.

Note, however, that these same relations can be obtained by considering the involutive braid operator fixed by $\tau(e_i \otimes e_j) = \mu^{-1}(e_j \otimes e_i)$ for $i < j$, and $\tau(e_i \otimes e_i) = e_i \otimes e_i$.

To derive the remainder of the Clifford algebra we use the right module structure in V and V', defined in [12], as well as the commutative pentagonal diagrams expressing the compatibility between the braidings extended from $V \otimes V$ to the spaces $V \otimes V'$, $V' \otimes V$ and $V' \otimes V'$, and the contraction map between V' and V. As explained in [12], such diagrams uniquely fix the corresponding extensions.

Let us denote by ρ the extended involutive braiding acting on $W \otimes W$, obtained by the above-mentioned procedure, so that

$$\begin{array}{lll}
\rho(e'_i \otimes e_j) = \mu(e_j \otimes e'_i) & \rho(e'_i \otimes e'_j) = \mu^{-1}(e'_j \otimes e'_i) & i < j \\
\rho(e'_i \otimes e_i) = e_i \otimes e'_i & \rho(e'_i \otimes e'_i) = e'_i \otimes e'_i & \\
\rho(e'_i \otimes e_j) = \mu^{-1}(e_j \otimes e'_i) & \rho(e'_i \otimes e'_j) = \mu(e'_j \otimes e'_i) & i > j.
\end{array} \tag{2}$$

We now introduce a consistent anti-multiplicative *-structure on our Clifford algebra, by requiring that ρ satisfy the sufficiency condition

$$(* \otimes *)\pi\rho = \rho(* \otimes *)\pi \tag{3}$$

where π is the standard permutation operator.

For this purpose we first generalize the algebra to a multiparametric one by means of the change $\mu \to \mu_{ij}$ in the braid relations; such that, for $i \neq j$ we have $\mu_{ij} = \mu_{ji}^{-1} = \exp(\mathrm{i}\theta_k)$, when $1 \leqslant k \leqslant (\nu - h - 1)(\nu - h)/2$ and $i, j \leqslant \nu - h$, and such that $\mu_{ij} = \mu_{ji}^{-1} = \exp(\mathrm{i}\lambda_k)$ with $1 \leqslant k \leqslant h - 1$ for $i, j \leqslant \nu - h + 1$, and finally $\mu_{ij} = \mu_{ji}^{-1} = \mu_k \in \mathbb{R}$, where $1 \leqslant k \leqslant \nu - h$ with $i \leqslant \nu - h, j \geqslant \nu - h + 1$ or $j \leqslant \nu - h, i \geqslant \nu - h + 1$. Here the index h denotes the number of negative terms in the signature of the metric of the classical pseudo-Euclidean space, associated with the isotropic basis $\{e_i, e'_i\}$. If we now define

$$\begin{array}{lll}
e_i^* = b_i e'_i & (e'_i)^* = b_i^{-1} e_i & i = 1, \ldots, \nu - h \\
e_i^* = \exp(\mathrm{i}\varphi_i) e_i & (e'_i)^* = \exp(-\mathrm{i}\varphi_i) e'_i & i = \nu - h + 1, \ldots, \nu
\end{array} \tag{4}$$

where $b_i, \varphi_i \in \mathbb{R}$, then it is easy to show that the generalized multiparametric braid relations are preserved. The choice (4) is clearly motivated by the observation that in the classical limit, $\lim \varphi_i \overset{\mu_{ij}\to 1}{\longrightarrow} 0$, $\lim b_i \overset{\mu_{ij}\to 1}{\longrightarrow} 1$, these relations reduce to the usual complex conjugation relations for the isotropic bases.

Furthermore, defining the *-action on the generators of the Clifford algebra by means of

$$(H(e_i))^* = (-1)^h H(e_i^*) \qquad (H(e'_i))^* = (-1)^h H(e_i'^*) \tag{5}$$

we also obtain in the classical limit ($\mu_{ij} \to 1$) the appropriate expression for the Hermitian adjoint operation on these generators. This *-structure is compatible with $Cl(\rho, W)$.

Let us now define a 'real' vector on W by the requirement $\boldsymbol{x}^* = \boldsymbol{x}^*$; it then follows from (4) that the components must satisfy

$$\begin{array}{lll}
(x^i)^* = b_i^{-1} x'^i & (x'^i)^* = b_i x^i & i = 1, \ldots, \nu - h \\
(x^i)^* = \exp(-\mathrm{i}\varphi_i) x^i & (x'^i)^* = \exp(\mathrm{i}\varphi_i) x'^i & i = \nu - h + 1, \ldots, \nu.
\end{array} \tag{6}$$

We can now use our quantum Clifford algebra to impose a non-commutative algebra with a consistent *-structure on the underlying quantum pseudo-Euclidean space, by requiring that the fundamental property of spinor transformations be preserved in the quantum case, i.e.

$$H(\boldsymbol{x})H(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{x}\rangle E. \tag{7}$$

where $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = x'^1 x^1 + \cdots + x'^\nu x^\nu$ is the fundamental quadric.

Under this assumption, and using the linearity of $H(\boldsymbol{x})$, it immediately follows that

$$\begin{aligned} x^i x^j &= m\bar{\tau}(x^i \otimes x^j) = \mu_{ij} x^j x^i \qquad & x'^i x^j &= m\tau(x'^i \otimes x^j) = \mu_{ij}^{-1} x^j x'^i \\ x^i x'^j &= m\tau(x^i \otimes x'^j) = \mu_{ij}^{-1} x'^j x^i \qquad & x'^i x'^j &= m\bar{\tau}(x'^i \otimes x'^j) = \mu_{ij} x'^j x'^i \end{aligned} \tag{8}$$

where m stands for the multiplication map. Note that $\langle \boldsymbol{x}, \boldsymbol{x}\rangle$ is central to this algebra, so it provides a sensible definition of length. Also note that (8) is consistent with the *-structure defined above. The involutive braid operator corresponding to the algebra (8) is obtained from ρ by making the exchange $\mu_{ij} \Leftrightarrow \mu_{ij}^{-1}$. From this it is evident that this new braid operator, which we denote by $\hat{R}$, will also satisfy the braid consistency relation. Moreover, defining

$$R := \pi\hat{R} \tag{9}$$

it can be shown that R is diagonal, and that it satisfies the quantum Yang–Baxter equation. We denote by $\mathcal{A}$ the non-commutative algebra of polynomials in the n variables $x^1, \ldots, x^\nu, x'^1, \ldots, x'^\nu$, and by $\mathcal{A}_{\hat{R}}$ the quotient algebra $\mathcal{A}/\mathcal{I}_{\hat{R}}$, where $\mathcal{I}_{\hat{R}}$ is the two-sided ideal in $\mathcal{A}$ generated by $(1 - \hat{R})(x \otimes x) = 0$. Thus $\mathcal{A}_{\hat{R}}$ is the twisted algebra of functions on the deformed n-dimensional vector space associated with the matrix $\hat{R}$.

The corresponding pure twisted general linear group, associated to the matrix R, follows readily from giving $\mathcal{A}_{\hat{R}}$ a comodule structure and applying the general formalism for quantum groups (see e.g. [13]). We thus obtain for the bialgebra $\mathcal{T}_R$, generated by the entries of the matrix t^α_β, the commutation relations

$$\mu_{\alpha\beta} t^\beta{}_\tau t^\alpha{}_\sigma = \mu_{\sigma\tau} t^\alpha{}_\sigma t^\beta{}_\tau \tag{10}$$

with

$$\mu_{\alpha\beta} = \begin{pmatrix} \mu_{ij} & \mu_{ij}^{-1} \\ \mu_{ij}^{-1} & \mu_{ij} \end{pmatrix}. \tag{11}$$

Now imposing our *-structure on the comodule action $\delta : \mathcal{A}_{\hat{R}} \to \mathcal{T}_R \otimes \mathcal{A}_{\hat{R}}$ by the requirement that δ is Hermitian, it is easy to compute the corresponding *-structure on the algebra $\mathcal{A}_{\hat{R}}$. In general, the consistency between the product and the *-structure is ensured by the construction.

Let us now assume that the algebra $\mathcal{T}_R$ is 'enlarged', by introducing the inverse of the corresponding quantum determinant (an alternative general and what we believe is a novel approach to this procedure is presented in detail in the following subsection). We shall denote this enlarged algebra by $\bar{\mathcal{T}}_R$.

Then it is possible to introduce the antipode map $\kappa : \bar{\mathcal{T}}_R \to \bar{\mathcal{T}}_R$, by requiring amtimultiplicativity, and

$$m(\kappa \otimes \mathrm{id})\phi(t^\alpha{}_\beta) = m(\mathrm{id} \otimes \kappa)\phi(t^\alpha{}_\beta) = \delta^\alpha{}_\beta. \tag{12}$$

The bialgebra $\bar{\mathcal{T}}_R$ becomes a deformed Hopf algebra with a compatible *-structure.

The components of the fundamental R-matrix associated with our quantum group are given by

$$R^{\alpha\beta}{}_{\sigma\tau} = \mu_{\beta\alpha} \delta^\alpha{}_\sigma \delta^\beta{}_\tau. \tag{13}$$

Comparing these expressions with those obtained by Schirrmacher [17] in his treatment of multiparametric deformations of $GL(n)$, it is clear that the matrix pseudo-group involved in our theory is a particular case of $GL_{X,q_{ij}}(n-h,h)$, with $X=1$ and the parameters related by (11). This result, of course, comes without surprise since our intertwiners were taken to be involutive to start with. Note, however, that the basis for our construction is completely different, since it hinges on the idea of utilizing our previously developed theory for a quantum Clifford algebra and adopting the ansatz (7) to induce the comodule structure $\mathcal{A}_{\hat{R}}$ for the 'coordinates'. Furthermore, as shown in the following subsection together with section 4, our previous analysis can be readily extended to the general construction of quantum analogues of linear (and orthogonal) groups, starting from general braid operators admitting abstract 'volume elements'.

2.1. General quantum determinants and associated quantum groups

In this subsection we shall abstract our previous analysis, and present a general construction of quantum analogues of linear (and orthogonal) groups, starting from the appropriate braid operators. We shall consider general (not necessarily involutive braidings) admitting abstract 'volume elements'. Conceptually, we follow [13], however in contrast to these papers we shall systematically use here the formalism of bicovariant bimodules [14], this allows us to derive the properties of the associated quantum determinants in a concise and elegant way.

In accordance with the notation introduced in the text and in [12], let Z be the vector space generated by the coordinates z^α. We shall denote by $R: Z^{\otimes 2} \to Z^{\otimes 2}$ the canonical braid operator (defining relations in the algebra of coordinates). Let us assume that $R: Z^{\otimes 2} \to Z^{\otimes 2}$ is such that there exists $n \in \mathbb{N}$ and an element $w_* \in Z^{*\wedge n}\backslash\{0\}$ such that $f \wedge \omega_* = 0$ for each $f \in Z^*$.

It is important to observe that $n \neq d = 2\nu = \dim(Z)$ in general (although in various interesting special cases the two numbers will coincide).

We shall also assume that the pairing between Z and Z^* is defined by

$$\langle z \otimes f\rangle = \langle R(z \otimes f)\rangle \tag{14}$$

where the symbol R will be used for all the braidings appearing in the braided monoidal category generated by Z, Z^* and the initial braiding. Finally, let us assume that this pairing is non-degenerate

Proposition 2.1. Under the above assumptions we have the following symmetry property:

$$\dim(Z^{\wedge k}) = \dim(Z^{\wedge(n-k)}). \tag{15}$$

In particular, $\dim(Z^{\wedge n}) = 1$ and $Z^{\wedge k} = \{0\}$ for $k > n$.

Proof. Let us consider the quantum Clifford algebra $Cl(Y)$ associated to $Y = Z \oplus Z^*$ and the corresponding braiding R. Furthermore, let us assume that $Z^{*\wedge}$ is embedded in $Z^{*\otimes}$ with the help of the inverse braiding $R^{-1}: Z^{*\otimes 2} \to Z^{*\otimes 2}$. The formulae

$$H(f)\psi = f \wedge \psi \qquad H(z)\psi = \iota_z \psi$$

where $f \in Z^*$ and $z \in Z$, define a representation H of $Cl(Y)$ in $Z^{*\wedge}$. Observing that $H(Z^*)\omega_* = \{0\}$ and applying the results from [12], it follows that there exists (the unique) injection $\rho: Z^\wedge \to Z^{*\wedge}$ intertwinning the corresponding representations and satisfying $\rho(1) = \omega_*$. In particular, $\rho(Z^{\wedge k}) \subseteq Z^{*\wedge n-k}$ and hence $\dim(Z^{\wedge k}) \leqslant \dim(Z^{*\wedge n-k}) = \dim(Z^{\wedge n-k})$. Hence ρ is bijective. $\square$

Let ω be the corresponding volume element in $Z^{\wedge n}$. Let $\mathcal{T}_R$ be the matrix bialgebra generated by abstract matrix elements $t^\alpha{}_\beta$ and the relations coming from the requirement that $R : Z^{\otimes 2} \to Z^{\otimes 2}$ intertwines $T \in M_d(\mathcal{T}_R)$.

By construction we have a natural coaction $T : Z \to \mathcal{T}_R \otimes Z$. This coaction map admits the unique unital multiplicative extension $T^\wedge : Z^\wedge \to \mathcal{T}_R \otimes Z^\wedge$. In particular, we have

$$T^\wedge(\omega) = \Delta \otimes \omega \tag{16}$$

where $\Delta \in \mathcal{T}_R$ is an element which will be called the quantum determinant. From the comodule property we find

$$\phi(\Delta) = \Delta \otimes \Delta \qquad \epsilon(\Delta) = 1.$$

Let us assume that $Z^\wedge$ is embedded in $Z^\otimes$, via the braiding R. We can write

$$\omega = \sum_\alpha s^\alpha \otimes z^\alpha \tag{17}$$

where $s^\alpha \in Z^{\wedge n-1}$.

Lemma 2.2. The elements s^α form a basis in $Z^{\wedge n-1}$.

Proof. This is a consequence of considerations contained in the previous proof. □

Hence, we can write $T^\wedge(s^\alpha) = \sum_\beta \bar{t}^\alpha{}_\beta \otimes s^\beta$, where $\bar{t}^\alpha{}_\beta$ satisfy $\phi(\bar{\tau}^\alpha{}_\beta) = \sum_\gamma \bar{t}^\alpha{}_\gamma \otimes \bar{t}^\gamma{}_\beta$ and $\epsilon(\bar{t}^\alpha{}_\beta) = \delta^\alpha{}_\beta$.

Let us consider a scalar matrix S given by

$$z^\beta \wedge s^\alpha = S^{\beta\alpha}\omega. \tag{18}$$

Lemma 2.3. The matrix S is invertible.

Proof. More generally, let us consider a pairing $[|] : Z^{\wedge k} \otimes Z^{\wedge n-k} \to \mathbb{C}$ given by

$$[\vartheta|\eta]\omega = \vartheta \wedge \eta.$$

It follows from the proof of proposition 2.1 that this pairing is non-degenerate. □

Now we can derive two algebraic relations between matrixes T and $\bar{T}$.

Proposition 2.4. We have

$$\Delta E = (\bar{T})^\sim T \tag{19}$$

$$\Delta E = T S(\bar{T})^\sim S^{-1}. \tag{20}$$

Proof. This is a direct consequence of (16), (17) and (18). □

Let $\bar{\mathcal{T}}_R$ be the algebra obtained by adding to $\mathcal{T}_R$ the formal inverse of Δ. It is easy to see that ϕ and ϵ admit natural extensions to $\bar{\mathcal{T}}_R$.

Proposition 2.5. (i) The matrix T is invertible in $M_d(\bar{\mathcal{T}}_R)$ and

$$T^{-1} = \Delta^{-1}(\bar{T})^\sim = S(\bar{T})^\sim S^{-1}\Delta^{-1}. \tag{21}$$

(ii) There exists the antipode map $\kappa : \bar{\mathcal{T}}_R \to \bar{\mathcal{T}}_R$, and in particular $\kappa(T) = T^{-1}$.

Proof. The statement (i) is a consequence of the previous proposition. The second statement follows from the observation that the relations defining $\bar{\mathcal{T}}_R$ are compatible with inverting Δ and T. □

The constructed quantum group $G \leftrightarrow \bar{\mathcal{T}}_R$ is the analogue of the general linear group. Note that such a construction allows the interpretation in terms of bicovariant bimodules. The formula

$$R(z^\alpha \otimes z^\beta) = \sum_\gamma (t^\beta{}_\gamma \circ z^\alpha) \otimes z^\gamma \tag{22}$$

consistently and uniquely determines a left $\bar{\mathcal{T}}_R$-module structure on Z so that the triple $(Z, \circ, T)$ determines a bicovariant bimodule Γ over $\bar{\mathcal{T}}_R$. Similarly, the formulae $a \circ 1 = \epsilon(a)1$ and $a \circ (\vartheta\eta) = (a^{(1)} \circ \vartheta)(a^{(2)} \circ \eta)$ determine left $\bar{\mathcal{T}}_R$-module structures on $Z^{\wedge,\otimes}$. The space Z is interpretable as the right-invariant part of $\Gamma \leftrightarrow Z \otimes \bar{\mathcal{T}}_R$, and T is the restriction on Z of the corresponding left action map $l_\Gamma : \Gamma \to \bar{\mathcal{T}}_R \otimes \Gamma$.

In particular,

$$T^\wedge(a \circ \vartheta) = \sum_k a^{(3)} c_k \kappa(a^{(1)}) \otimes (a^{(2)} \circ \vartheta_k) \tag{23}$$

where $\sum_k c_k \otimes \vartheta_k = T^\wedge(\vartheta)$. Furthermore, we have

$$a \circ \omega = \omega\lambda(a) \tag{24}$$

where $\lambda : \bar{\mathcal{T}}_R \to \mathbb{C}$ is a (non-trivial) linear multiplicative functional. This fact can be used to derive a simple commutation relation between Δ and elements of $\bar{\mathcal{T}}_R$. Let $D : \bar{\mathcal{T}}_R \to \bar{\mathcal{T}}_R$ be an automorphism given by

$$D = (\mathrm{id} \otimes \lambda)\mathrm{ad} \tag{25}$$

where $\mathrm{ad} : \bar{\mathcal{T}}_R \to \bar{\mathcal{T}}_R \otimes \bar{\mathcal{T}}_R$ is the corresponding adjoint action, explicitly given by $\mathrm{ad}(a) = a^{(2)} \otimes \kappa(a^{(1)})a^{(3)}$.

Lemma 2.6. We have

$$\Delta a = D^{-1}(a)\Delta \tag{26}$$

for each $a \in \bar{\mathcal{T}}_R$.

Proof. A direct computation gives

$$T^\wedge(a \circ \omega) = \Delta \otimes \omega\lambda(a) = a^{(1)}\Delta\kappa(a^{(3)}) \otimes \omega\lambda(a^{(2)}).$$

In other words $\Delta\lambda(a) = a^{(1)}\Delta\kappa(a^{(3)})\lambda(a^{(2)})$. Equivalently, (26) holds. □

To conclude this subsection, let us analyse the quantum determinant Δ of the orthogonal subgroups of G. Let us assume that the space Z is endowed with a (not necessarily positive) scalar product (,) such that $(\omega, \omega) \neq 0$. Here we have assumed that (,) is naturally extended to $Z^\otimes$.

Let $\mathcal{C}$ be the Hopf algebra obtained from $\bar{\mathcal{T}}_R$ by requiring the invariance of (,) under $T^{\otimes 2}$. This $\mathcal{C}$ represents the corresponding orthogonal group. We shall denote by the same symbols the projected entities.

Lemma 2.7. We have

$$\Delta^2 = 1. \tag{27}$$

Proof. Applying the invariance condition we find $(\omega, \omega) \otimes 1 = (\omega, \omega) \otimes \Delta^2$, and hence (27) holds. □

Thus, we can write

$$\Delta = P_+ - P_- \qquad P_\pm = (1 \pm \Delta)/2$$

where

$$P_-^2 = P_- \qquad P_+^2 = P_+ \qquad P_- P_+ = P_+ P_- = 0$$

in the framework of $\mathcal{C}$. Let $J \subseteq \mathcal{C}$ be the set consisting of elements b satisfying

$$\Delta b = b \Leftrightarrow P_- b = 0.$$

The set J is a co-ideal and a two-sided ideal in $\mathcal{C}$. Moreover, $\kappa(J) \subseteq J$, and hence it is possible to factorize through J. Geometrically, this corresponds to passing to the normal subgroup consisting of unimodular matrices.

2.2. $O_\mu(2\nu - h, h)$ groups

We shall now require that the fundamental quadric $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$, which was previously shown to be central to the algebra of 'coordinates', should be invariant under the co-action δ. Thus we must have

$$\tilde{T} \begin{pmatrix} 0 & I_\nu \\ I_\nu & 0 \end{pmatrix} T = \begin{pmatrix} 0 & I_\nu \\ I_\nu & 0 \end{pmatrix} \tag{28}$$

where I_ν is the identity matrix in ν-dimensions.

Moreover, making use of the bicovariant bimodule interpretation (22) explained in subsection 2.1 and applying it to our specific involutive braiding determined by equations (8), we have

$$\hat{R}(x^i \otimes x^j) = \mu_{ij} x^j \otimes x^i = \sum_\alpha (t^j{}_\alpha \circ x^i) \otimes x^\alpha \Rightarrow t^j{}_\alpha \circ x^i = \mu_{ij} \delta^j{}_\alpha x^i \tag{29}$$

$$\hat{R}(x'^i \otimes x^j) = \mu_{ij}^{-1} x^j \otimes x'^i = \sum_\alpha (t^j{}_\alpha \circ x'^i) \otimes x^\alpha \Rightarrow t^j{}_\alpha \circ x'^i = \mu_{ij}^{-1} \delta^j{}_\alpha x'^i \tag{30}$$

$$\hat{R}(x^i \otimes x'^j) = \mu_{ij}^{-1} x'^j \otimes x^i = \sum_\alpha (t^{j'}{}_\alpha \circ x^i) \otimes x^\alpha \Rightarrow t^{j'}{}_\alpha \circ x^i = \mu_{ij}^{-1} \delta_\alpha{}^{j'} x^i \tag{31}$$

$$\hat{R}(x'^i \otimes x'^j) = \mu_{ij} x'^j \otimes x'^i = \sum_\alpha (t^{j'}{}_\alpha \circ x'^i) \otimes x^\alpha \Rightarrow t^{j'}{}_\alpha \circ x'^i = \mu_{ij} \delta_\alpha{}^{j'} x'^i. \tag{32}$$

Consequently,

$$\begin{aligned} t^\beta{}_\alpha \circ \omega = \sum_{\alpha_1 \ldots \alpha_{2\nu}}^{2\nu} & (t^\beta{}_{\alpha_{2\nu}} \circ x'^\nu) \wedge \cdots \wedge (t^{\alpha_\nu}{}_{\alpha_{\nu-1}} \circ x'^1) \wedge (t^{\alpha_\nu}{}_{\alpha_{\nu-1}} \circ x^\nu) \wedge \cdots \wedge (t^{\alpha_1}{}_\alpha \circ x'^1) \\ & = \omega \delta^\beta{}_\alpha. \end{aligned} \tag{33}$$

This in turn implies that $\lambda(T) = I$. Furthermore, since λ is a homomorphism of algebras, $\lambda(\kappa(T))\lambda(T) = I$, i.e. $\lambda(\kappa(T)) = \lambda(T)^{-1} = \lambda(T) = I$. Hence

$$D(t^\beta{}_\alpha) = \lambda(t^\beta{}_\gamma)\lambda(\kappa(t^\gamma{}_\delta)) \otimes (\lambda \otimes \mathrm{id})\mathrm{ad}(t^\delta{}_\alpha) = t^\beta{}_\alpha. \tag{34}$$

It then follows from lemma 2.6 that (28) implies that the determinant of our quantum matrices is central, while lemma 2.7 shows that this determinant Δ is subject to the additional restriction $\Delta^2 = 1$.

Finally, it also follows from (28) that

$$\kappa(T) = \begin{pmatrix} 0 & I_\nu \\ I_\nu & 0 \end{pmatrix} \tilde{T} \begin{pmatrix} 0 & I_\nu \\ I_\nu & 0 \end{pmatrix}. \tag{35}$$

3. Twisted spin groups associated to 2ν-(Pseudo)-Euclidean spaces

In the preceding section we have developed a *-structure compatible algebra for the twisted groups of proper and improper rotations in (pseudo)-Euclidean spaces of arbitrary even dimensions and arbitrary signatures of the metric. Our formalism resulted from first considering a quantum Clifford algebra based on an involutive braid operator. By requiring that the fundamental property of spinor transformations—given by equation (7), be preserved, we obtained an algebra $\mathcal{A}_{\hat{R}}$ for the *coordinates* of the underlying (pseudo)-Euclidean spaces. We then constructed a twisted matrix algebra $\mathcal{T}_R$ from the comodule structure of $\mathcal{A}_{\hat{R}}$. By further restricting the resulting pseudo-group to leave invariant the fundamental quadric $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$, we arrived at our desired results.

The program we propose to develop in this section beings also with the multiparametric quantum Clifford algebra $Cl(W, \rho)$, together with the preservation of the property (7) for the spinor transformations and the resulting deformed algebra (8) for the coordinates. However, instead of considering the quantum matrix group which co-acts on the quantum plane generated by these coordinates, we shall construct the quantum groups associated directly to spinor transformations, i.e. the quantum $\mathrm{Spin}_\mu(2\nu)$ groups. We consider it important at this point to stress the fact that even though $\mathrm{Spin}_\mu(4)$ groups have been considered previously in the literature [9–11], and even though our results for $\nu = 2$ agree with those of some of the referred authors, our formalism, based on quantum Clifford algebras and quantum spinor spaces, allows for a general consideration of quantum spin groups for any dimensions and signatures of the underlying (pseudo)-Euclidean spaces. It is also important to mention that our constructions are applicable to the general (non-involutive) braid operators. However, some constructions and concrete computations will be worked out in the context of the braidings given by ρ.

To begin, recall [1] that the quantum Clifford product is uniquely determined by the relations

$$H_1 \cdot 1 = e_i \qquad H_i \cdot e_j = e_i \wedge e_j \qquad H_I' \cdot e_j = \iota_{e_i'} e_j = e_i'(e_j) = \delta_{ij} \tag{36}$$

where $\{e_i\}$ is the isotropic basis in V, introduced in section 2, $\{e_i'\}$ is the reciprocal basis in V', and $H_i \equiv H(e_i)$, $H_i' \equiv H(e_i')$. We can then define a quantum spinor by

$$\xi = \sum_{p=0}^{\nu} \sum_{k_1 < \cdots < \kappa_p} \xi^{k_1 \ldots k_p} \otimes H_{k_1} \ldots H_{k_p} \cdot 1 \tag{37}$$

where the 2^ν components $\xi^{k_1 \ldots k_p}$ are the generators of a non-commutative free algebra $\mathcal{S}$, and the symbol $\sum_{k_1 < \cdots < k_p}$ is to be interpreted as no sum in the case $p = 0$, so $\xi^{k_1 \ldots k_p} = \xi^0$ when $p = 0$.

Note that in the classical limit $\mu_{ij} \to 1$, the above expression reduced to the usual definition of a spinor as an element in the graded Grassmann algebra of the basis vectors in the corresponding (pseudo)-Euclidean space to which the spinor is associated.

We can introduce a bilinear inner product on the quantum spinor spaces $\mathcal{S}$ by first defining the involutive and anti-multiplicative T-transpose operation, $\xi \in S \to \xi^{\mathrm{T}} \in S'$, which maps linearly spinors in S to spinors in the dual space S'. This operation is uniquely defined by its action on the generators of the Clifford algebra:

$$1^{\mathrm{T}} = 1' \qquad H_i^{\mathrm{T}} = H_i' \qquad (H_i H_j)^{\mathrm{T}} = H_j' H_i'. \tag{38}$$

Hence the T-transpose operation maps Clifford product from the left to Clifford product action from the right, and

$$\xi^{\mathrm{T}} = \sum_{p=0}^{\nu} \sum_{k_1 < \cdots < k_p} \xi^{k_1 \ldots \kappa_p} \otimes 1' \cdot H'_{k_p} \ldots H'_{k_1}. \tag{39}$$

Note that by virtue of (36) and (38) the elements $\{(H_{k_1} \ldots H_{k_p} \cdot 1)^{\mathrm{T}} = 1' \cdot H'_{k_p} \ldots H'_{k_1}\}$ form a basis reciprocal to $\{(H_{k_1} \ldots H_{k_p} \cdot 1\}$, $k_1 < \cdots < k_p$, which allows us to define a scalar product for homogeneous spinors of p-degree, given by

$$\begin{aligned}
[(\xi^{(p)})^{\mathrm{T}}, \eta^{(p)}] :&= \sum_{k_1 < \cdots < k_p} \xi^{k_1 \ldots k_p} \eta^{k_1 \ldots k_p} \otimes [1' \cdot H'_{k_p} \ldots H'_{k_1}, H_{k_1} \ldots H_{k_p} \cdot 1] \\
&= \sum_{k_1 < \cdots < k_p} \xi^{k_1 \ldots k_p} \eta^{k_1 \ldots k_p} \otimes 1' \cdot H'_{k_p} \ldots H'_{k_1} H_{k_1} \ldots H_{k_p} \cdot 1 \\
&= \sum_{k_1 < \cdots < k_p} \xi^{k_1 \ldots k_p} \eta^{k_1 \ldots k_p} \otimes 1.
\end{aligned} \tag{40}$$

Requiring that the scalar product of any two spinors respects gradation, we thus have

$$[\xi^{\mathrm{T}}, \eta] = \sum_{p=0}^{\nu} \sum_{k_1 < \cdots < k_p} \xi^{k_1 \ldots k_p} \eta^{k_1 \ldots k_p}. \tag{41}$$

Also, in analogy to the classical Cartan spinor theory, we can define a fundamental spinor bilinear by means of

$$(\xi, \xi) = [\xi^{\mathrm{T}}, C \cdot \xi]. \tag{42}$$

Here C is a spinor metric operator given by

$$\begin{aligned}
C = \sum_{p=0}^{\nu} &(-1)^{(\nu-p)(\nu-p+1)/2} \\
&\times \sum_{\substack{\pi \in S_\nu \\ \pi(1) < \cdots < \pi(p) \\ \pi(p+1) < \cdots < \pi(\nu)}} (-1)^{l(\pi)} a_{\pi(1) \ldots \pi(p)}(\mu) H_{\pi(1)} \ldots H_{\pi(p)} (H_{\pi(p+1)} \ldots H_{\pi(\nu)})^{\mathrm{T}}
\end{aligned} \tag{43}$$

with $l(\pi) =$ length of the permutation π and, for the case of pure twists,

$$\begin{aligned}
a_{\pi(1) \ldots \pi(p)} &= [\mu_{\langle \pi(1)\pi(\nu) \rangle} \ldots \mu_{\langle \pi(1)\pi(p+1) \rangle} \ldots \mu_{\langle \pi(p)\pi(\nu) \rangle} \ldots \mu_{\langle \pi(p)\pi(p+1) \rangle}]^{1/2} \\
a_{\pi(1) \ldots \pi(\nu)} &= a_0 = 1
\end{aligned} \tag{44}$$

where the symbol $\langle\,\rangle$ denotes pair ordering of indices so that the first one is lower than the second.

It is easy to verify that with (43)

$$C^{\mathrm{T}} = (-1)^{\nu(\nu+1)/2} C \tag{45}$$

and

$$(\xi, \xi)^{\mathrm{T}} = (-1)^{\nu(\nu+1)/2} (\xi, \xi) \tag{46}$$

which is the quantum analogue of polarity of the spinor bilinear (42).

Making use of (43) it can be readily shown that (42) may be written as

$$(\xi, \xi) = \sum_{p=0}^{\nu} (-1)^{(\nu-p)(\nu-p+1)/2} \sum_{\substack{\pi(1) < \cdots < \pi(p) \\ \pi(p+1) < \cdots < \pi(\nu)}} (-1)^{l(\pi)} a_{\pi(1) \ldots \pi(p)} \xi^{\pi(1) \ldots \pi(p)} \xi^{\pi(p+1) \ldots \pi(\nu)}. \tag{47}$$

Now, taking as a basis the 2^ν elements of the quantum Clifford algebra $\{1, H_{k_1}\ldots H_{k_p}|k_1 < k_2 < \cdots < k_p, p = 1, 2, \ldots, \nu\}$ ordered in such a way that those with an even number of indices and in an increasing degree sequence come first, followed by those elements with an odd number of indices also in an increasing degree sequence, we can write a spinor as a column vector where the first $2^{\nu-1}$ entries correspond to a semi-spinor of the first type (which we shall denote by φ), while the last $2^{\nu-1}$ entries correspond to a semi-spinor of the second type (which we shall denote by ψ), in Cartan's terminology.

It is evident from (47) that the fundamental spinor bilinear involves products of components of semi-spinors of the same type if $\nu =$ even while if $\nu =$ odd the products are of semi-spinors of the two different types.

In the classical Cartan spinor theory, the action of the operator $H(\boldsymbol{x}) = \sum_{i=1}^{\nu}(x^i H_i + x'^i H_i')$ on spinors, with $\boldsymbol{x}$ a unit vector, corresponds to a reflection in the hyperplane perpendicular to $\boldsymbol{x}$. A proper rotation on vectors then corresponds to an even product of Clifford operators acting on spinor space. Thus, relative to the above-described basis, the spin group matrices B are block diagonal.

Taking the entries of such a matrix as the generators of the free algebra $\mathcal{B}$ of non-commutative polynomials, and requiring that $\mathcal{B}$ satisfies the connection axiom

$$\phi(a\cdot b) = \phi(a)\cdot\phi(b) \qquad \epsilon(a\cdot b) = \epsilon(a)\cdot\epsilon(b) \qquad \forall a, b \in \mathcal{B} \tag{48}$$

equips $\mathcal{B}$ with a bialgebra structure.

Furthermore, since our quantum Clifford algebra involves an involutive multiparametric braid, it is natural to expect such a braiding for $\mathcal{B}$. Imposing the requirement that $\det_q B = 1$ we have, making use of the results of Schirrmacher [17], that

$$b^\alpha{}_\beta b^\lambda{}_\sigma = \frac{q_{\alpha\lambda}}{q_{\beta\sigma}} b^\lambda{}_\sigma b^\alpha{}_\beta \tag{49}$$

where, because of the block-diagonal structure of the matrix $\mathcal{B}$, the indices $\alpha, \beta, \lambda, \sigma \in \mathbb{Z}$ take values ranging from 1 to $2^{\nu-1}$ or from $2^{\nu-1}+1$ to 2^ν, and

$$q_{\alpha\lambda} = q_{\lambda\alpha}^{-1} \qquad \text{if } \alpha \leqslant \lambda. \tag{50}$$

We can then construct a quantum matrix algebra by considering the quotient algebra $\mathcal{B}_R = \mathcal{B}/\mathbb{I}_R$ by the two-sided ideal $\mathbb{I}_R$ generated by the braid type relations $R(B\otimes I)(I\otimes B) = (I\otimes B)(B\otimes I)R$.

Moreover, from the above-constructed involutive braid operator for the $\mathcal{B}$ matrices, we can obtain the associated quotient algebra $\mathcal{S}_{\hat{R}} = \mathcal{S}/\mathbb{I}_{\hat{R}}$, where $\mathbb{I}_{\hat{R}}$ is the two-sided ideal in $\mathcal{S}$ generated by $(1-\hat{R})(\xi\otimes\xi) = 0$ and, as before, $\hat{R} = \pi R$. Since $(1-\hat{R})(\delta(\xi)\otimes\delta(\xi)) = 0$, $\mathcal{S}_{\hat{R}}$ acquires a comodule structure with a co-action map

$$\delta : \mathcal{S}_{\hat{R}} \to \mathcal{B}_R \otimes \mathcal{S}_{\hat{R}} \tag{51}$$

defined by

$$\delta(\xi^\alpha) = \sum_{\beta=1}^{2\nu} b^\alpha{}_\beta \otimes \xi^\beta. \tag{52}$$

To obtain the different quantum spin groups, we need to impose additional constraints on $\mathcal{B}_R$ which, as suggested by the classical spinor theory, should be determined by the fundamental spinor bilinear (42). Thus we shall require that

(i) the spinor bilinear (ξ, ξ) be central relative to the algebra $\mathcal{S}_{\hat{R}}$;

(ii) the quantum determinant of each block in the matrix B should be central and unimodular;

(iii) the spinor bilinear should be invariant under the coaction map δ. In other words $\delta : (\xi, \xi) \to 1 \otimes (\xi, \xi)$. This in turn implies

$$b^{\alpha}{}_{\lambda} b^{\beta}{}_{\sigma} C_{\alpha\beta} = C_{\lambda\sigma} \tag{53}$$

where $C_{\alpha\beta}$ are the matrix elements of the spinor metric in terms of compound indices determined by (44), (47) and the specific ordering described previously. Note from (53) that since the matrix C is invertible, $\kappa(B) = C^{-1}\tilde{B}C$.

For the final axiom we need to consider the analogue of the classical homomorphism property, relating spin groups and (pseudo)-Euclidean groups. For this purpose let us first define the 'spacetime coordinates' as

$$x_{\alpha} = (\xi, H_{\alpha}\xi) = \xi^{\mathrm{T}} C H_{\alpha} \xi. \tag{54}$$

The indices of these 'coordinates' are lowered and raised with the metric of the fundamental quadric $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$. Hence we can rewrite (54) as

$$x^{\alpha} = \xi^{\mathrm{T}} C H_{\alpha}^{\mathrm{T}} \xi. \tag{55}$$

Co-acting with $\mathcal{B}_R$ on (55) according to (52) we have

$$\delta : \xi^{\alpha} \to b^{\sigma}{}_{\lambda} C_{\sigma\beta} b^{\gamma}{}_{\mu} (H_{\alpha}^{\mathrm{T}})_{\beta\gamma} \otimes \xi^{\lambda} \xi^{\mu} \tag{56}$$

and making use of (53) we obtain

$$\delta : \xi^{\alpha} \to C_{\lambda\sigma} (\kappa(B))^{\sigma}_{\beta} (H_{\alpha}^{\mathrm{T}})_{\beta\gamma} b^{\gamma}{}_{\mu} \otimes \xi^{\lambda} \xi^{\mu}. \tag{57}$$

Thus,

(iv) assume that the vector space of generators H_{α} is invariant under the constructed adjoint action δ.

Axiom (iv) allows us to define the quantum matrix elements $t^{\alpha}{}_{\beta}$ by

$$\kappa(B) H_{\alpha}^{\mathrm{T}} B = t^{\alpha}{}_{\beta} H_{\beta}^{\mathrm{T}} \tag{58}$$

so

$$\begin{aligned} \delta : \xi^{\alpha} \to t^{\alpha}{}_{\beta} \otimes \xi^{\beta} &= t^{\alpha}{}_{\beta} C_{\lambda\rho} (H_{\beta}^{\mathrm{T}})_{\rho\mu} \otimes \xi^{\lambda} \xi^{\mu} \\ &= t^{\alpha}{}_{\beta} \otimes (\xi, H_{\beta}^{\mathrm{T}} \xi). \end{aligned} \tag{59}$$

The geometrical meaning of (58) is that after restricting the adjoint action on the space of generators H_{α}, we should obtain the standard action of the quantum orthogonal group (as in the classical theory). It is worth mentioning that in lower dimensions such invariance holds automatically.

These axioms are sufficient to determine univocally and consistently the quantum spinor algebra and quantum spin groups since they give the quantum parameters in (49) in terms of the μ's of our Clifford algebra. However, they may not all be necessary. In fact, note in particular that the centrality of the quadratic form (ξ, ξ) can be *derived* from the other conditions, so it is not actually an axiom but a consequence of the general property that every left-invariant element in a braided-symmetric algebra, built over the right-invariant part of an arbitrary bicovariant bimodule, is automatically central. In more detail, using conditions (ii)–(iv) and applying the general methods for constructing Hopf algebras via intertwiner-type relations, we end up with a Hopf algebra $\mathcal{B}'$ based on the matrix B. The later describes the co-action δ on the spinor vector space S:

$$\delta(\xi^i) = \sum_j b^i_j \otimes \xi^j.$$

Furthermore, it can be shown that the space S is equipped with a natural left-module structure $\circ$ over $\mathcal{B}'$, so that S is interpretable as a right-invariant part of a bicovariant bimodule Ξ.

Now, it is sufficient to observe that the braid operator generating the algebra $\mathcal{S}_{\hat{R}}$ is precisely the braiding intrinsically associated to Ξ. In other words, the braiding is $\xi^i \otimes \xi^j \mapsto \sum_k (b^i_k \circ \xi^j) \otimes \xi^k$, and the property (i) easily follows.

We have included (i) in the list of conditions because of its importance in practical calculations (particularly for higher-dimensional spaces).

Finally, let us observe that the following interesting property holds. If we particularize the considerations to the involutive braidings we considered, then it turns out that contragradient 'coordinates', expressed in terms of (55), satisfy the commutation relations (8) with $\mu_{ij} \to \mu_{ij}^{-1}$. Furthermore, a more detailed analysis shows that it is not possible to avoid consistently this '2-periodicity'. However, as we shall see, our construction still gives the appropriate quantum orthogonal group as a homomorphic image of the quantum spin group. It is also worth noticing that a similar phenomena appears in our general theory of quantum Clifford algebras [12], where Clifford algebras associated to coordinates and derivatives are related by the same kind of transformation of the deformation parameters.

To further clarify the above remarks, note that multiplying (58) by itself from the right and from the left, adding the results, and making use of the Clifford algebra $Cl(\rho, W)$ with $\mu_{\alpha\beta}$ given by (11), we can write

$$\begin{aligned}\kappa(B)(H_\alpha^{\mathrm{T}} H_\beta^{\mathrm{T}} + \mu_{\alpha\beta}^{-1} H_\beta^{\mathrm{T}} H_\alpha^{\mathrm{T}})B &= \sum_{i,j} (t^\alpha{}_i t^\beta{}_j - \mu_{\alpha\beta}^{-1}\mu_{ij} t^\beta{}_j t^\alpha{}_i) H_i' H_j' \\ &\quad + \sum_{i,j} (t^\alpha{}_{i'} t^\beta{}_{j'} - \mu_{\alpha\beta}^{-1}\mu_{ij} t^\beta{}_{j'} t^\alpha{}_{i'}) H_i H_j + \sum_{i,j} (t^\alpha{}_i t^\beta{}_{j'} - \mu_{\alpha\beta}^{-1}\mu_{ij}^{-1} t^\beta{}_{j'} t^\alpha{}_i) H_i' H_j \\ &\quad + \sum_{i,j} (t^\alpha{}_{i'} t^\beta{}_j - \mu_{\alpha\beta}^{-1}\mu_{ij}^{-1} t^\beta{}_j t^\alpha{}_{i'}) H_i H_j'. \end{aligned} \tag{60}$$

Furthermore, using $Cl(\rho, W)$, the left-hand side of (60) becomes LHS $= (\delta^\alpha{}_l \delta^\beta{}_{l'} + \delta^\alpha{}_{l'} \delta^\beta{}_l)E$. It clearly follows then that the $t^\alpha{}_\beta$ on the RHS must satisfy (10) with the replacement $\mu_{\alpha\beta} \to \mu_{\alpha\beta}^{-1}$, together with

$$\sum_i (t^\alpha{}_i t^\beta{}_{i'} + t^\alpha{}_{i'} t^\beta{}_i) = \delta^\alpha{}_l \delta^\beta{}_{l'} + \delta^\alpha{}_{l'} \delta^\beta{}_l. \tag{61}$$

This last result is equivalent to (28), so invariance of the fundamental spinor bilinear implies invariance of the quadric $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$. Thus we have established the homomorphism of our quantum spin groups with the SO_μ groups.

In order to be able to account for the different possible signatures of the underlying (pseudo)-Euclidean spaces, we need to introduce a compatible *-structure for the algebras $\mathcal{S}_{\hat{R}}$ and $\mathcal{B}_R$. Such a *-structure can be readily obtained by recalling the *-structure that we derived for the 'coordinates' in the preceding section (cf equation (6)), making use of (55) which relates 'coordinates' to spinor components, and requiring that $(\delta(\xi^{i_1,\dots,i_p}))^* = \delta((\xi^{i_1,\dots,i_p})^*)$.

Note that our construction involves expressing univocally the $2^{\nu-1}(2^\nu - 1)$ parameters $q_{\alpha\beta}$ in the quantum spin matrices in terms of the $(1/2)\nu(\nu-1)$ parameters of the Clifford algebra. Except for the case $\nu = 2$, this is a highly overdetermined and non-trivial problem, which is solved by applying the axioms (i)–(iv) above. We have concentrated in the appendix the results for $\nu = 2$, both for the Euclidean and Lorentzian metrics, as well as for the cases $\nu = 3, 4$ (also for Euclidean and Lorentzian metrics) which further illustrates our construction and, as pointed out in the introduction, may be the most relevant to the deformation of physics theories in higher dimensions and deformed twistor theory.

Due to the relative complexity involved in solving for the parameters in the quantum spin matrices in terms of those in the original Clifford algebra, in particular for the cases of higher-dimensional spaces, it is worthwhile to consider the possibility of alternative approaches which might lead to a simplification of our calculations and constructions. One such possible approach originates from the works of Drinfel'd and Reshetikhin [15] (cf also the related discussion in [18]) which allows one to obtain multiparametric quantum groups by 'twisting' arbitrary q-groups with a diagonal matrix $F = \mathrm{diag}\,(f_{11}, f_{12}, \ldots, f_{nn})$, subject to the condition $f_{ij} f_{ji} = 1$. The geometrical meaning of these quantum group twists has been investigated by Chaichian and Demichev [10] who showed that twists may be understood as q-deformed coordinate transformations by means of an auxiliary algebra of q-deformed n-beins. Thus, in particular, we could consider starting our formalism with ordinary commutative objects and classical groups, and try to arrive at our final constructions and pure twisted groups via this technique.

Specifically, by virtue of the fact that (7) is frame independent, a quantum transformation of coordinates using the q-beins implies immediately a deformation of the Clifford algebra to $Cl(\rho, W)$. We could then indeed start with commuting coordinates and the classical Clifford algebra, and using Cartan's formalism in such a frame-independent fashion attempt to derive the quantum spin groups in an alternative and perhaps simpler manner. In fact, we have essentially followed such a construction in [19], using the well known fact that the elements of the spin group are the even elements of the Clifford algebra. Thus proper rotations are given by operators of the form $B = H(\boldsymbol{x})H(\boldsymbol{y})$ and their products. Clearly B may be calculated in the frames of commuting coordinates and the classical Clifford algebra, and perform the deformations afterwards by means of the q-beins, to get the associated deformed matrix. Note, however, that these rotation operators co-act on spinors and not on the vector space of the generators of the algebra of the coordinates. Consequently, the corresponding twisted matrices are expressed in terms of the spinor basis generated by the Clifford algebra (cf equation (37)), with the ordering described in the paragraph following equation (47). This means, in particular, that the analysis in [10] cannot be used directly to infer that these twisted spin group matrices satisfy the axioms of deformed Hopf algebras and that they can be associated to a quantum group. As it turned out, the entries in the resulting B matrix have commutation relations determined by the algebra of the coordinates (8), extended to apply to coordinates of different vectors. This algebra of B does not satisfy the usual axiom for the coproduct and, therefore, does not lead to a quantum group. We did show in fact in [19] that, working in the context of braided categories and specifically using the same braid of the coordinates for the braid of the coproduct, one could interpret the twisted B matrices as being actually elements of a braided spin group and not a quantum group. For further details on this approach, we refer the reader to the above-cited paper. As a final remark on this issue, we note that although we could still apply the q-bein technique to the classical block diagonal matrices of the spin group to get an algebra for the corresponding twisted quantum group, such a procedure would not be of much help in simplifying the calculations needed to relate the parameters of the resulting twisted group with those of the Clifford algebra. To obtain this relation it is essential to use axioms (iii) and (iv) of our construction, which cannot be derived by the q-bein technique.

On the other hand, the use of the technique of Drinfel'd and Reshetikhin [15], would be important for the construction of quantum spin groups from non-involutive Clifford algebra braids, along the lines proposed in the next section.

4. Quantum spin groups from non-involutive braids

As mentioned in the introduction, involutive braids although simple, are not trivial. From a mathematical point of view very interesting purely quantum phenomena already appear at the level of diagonal R-matrices, as for example possible deviations, from its classical counterpart, in the Poincaré series of the braided exterior algebra. On the other hand, it is possible that such milder deformations might turn out to be physically less interesting.

This, however, does not pose a major limitation to our approach since, as we also have mentioned previously, the essential concepts of our formalism should apply equally well to more general braidings. Thus, in order to complete the discussion, we shall concentrate in this last section in providing an outline of the steps that would have to be followed to extend our procedure to more general braidings associated with our general theory of quantum Clifford algebras. (The intermediate steps are very much suggested by following the presentation in sections 2 and 3.) For this purpose, we recall first that the braiding in our quantum Clifford algebra [12] is given by the block matrix

$$\rho = \begin{pmatrix} \mu^{-2}\sigma & 0 & 0 & 0 \\ 0 & 0 & \sigma^{-1} & 0 \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & 0 & \mu^{-2}\sigma \end{pmatrix}. \tag{62}$$

In the above matrix, the operator σ has been extended from $V \otimes V$ to the braiding on the direct sum $W = V \oplus V'$. This extension is fixed uniquely and consistently by requiring the functoriality of the corresponding contraction maps, as explained in the above referred paper. The blocks of the extension are denoted by the same symbol σ, a notation which should not lead to any confusion as the σ's are uniquely fixed by specifying the domains. This construction implies immediately that the operator ρ satisfies the braid equation.

Another way to verify that (62) satisfies the braid equation is by directly considering the action of both sides of that expression on all possible triple tensor products of the subspaces V and V', with ρ given by (62), and then applying linearity. Furthermore, using (7) again as an ansatz we arrive at the $\hat{R}$ matrix

$$\hat{R} = \begin{pmatrix} \mu^{2}\tilde{\sigma}^{-1} & 0 & 0 & 0 \\ 0 & 0 & \tilde{\sigma} & 0 \\ 0 & \tilde{\sigma}^{-1} & 0 & 0 \\ 0 & 0 & 0 & \mu^{-2}\sigma^{-1} \end{pmatrix} \tag{63}$$

which, by construction, also satisfies the braid relation. This implies that the universal matrix $R = \pi\hat{R}$ obeys the Yang–Baxter equation, which guarantees consistency of the 'RTT' equations and which, in turn, define a quantum $GL(n)$ group.

As a next step we impose the sufficiency condition (3) in order to introduce the concept of reality as well as a consistent *-structure for R and for the Clifford and 'coordinate' algebras. In addition, and having in mind higher-dimensional spaces, we make use of the technique of Drinfel'd and Reshetikhin [15], to obtain a multiparametric $R^{(F)}$-matrix from our one-parameter R by 'twisting' with a unitary (so as to respect the *-structure) matrix F.

From this stage on we would only have to follow, in principle and with the appropriate modifications, the steps detailed in sections 2 and 3 to arrive at the different quantum spin groups. More specifically, we would need to use the lemmas in section 2.1, applied to our new braidings, to verify that the requirement of invariance under the coaction map δ of the central fundamental quadric $\langle \boldsymbol{x}, \boldsymbol{x}\rangle$, leads to centrality of the determinant Δ and to the additional restriction $\Delta^2 = 1$. Finally, we would follow the procedure in section 3 to derive the commutation relations for the spin matrices, and use the axioms (i)–(iv) to reduce the number of parameters occurring in these commutation relations, and relate them to the ones involved in the Clifford algebra. Axiom (iv), in particular, would help to establish as well the homomorphism with the quantum $SO(2\nu)$ groups.

The details of the above program are part of an ongoing research program by the authors and will be presented separately.

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Appendix. Quantum $\mathrm{Spin}_\mu(2\nu - h, h), \nu = 2, 3, 4$

A.1. Quantum $\mathrm{Spin}_\mu(4 - h, h)$ *groups*

In this case the spinor metric operator (43) becomes

$$C = H_1 H_2 - \sqrt{\mu} H_1 H_2' + \sqrt{\mu} H_2 H_1' - H_2' H_1' \tag{64}$$

and a quantum spinor, in the ordering described above, is given by

$$\xi = \phi^1 + \psi^1 H_1 + \psi^2 H_2 + \phi^2 H_1 H_2. \tag{65}$$

Hence the fundamental spinor bilinear has the form

$$\xi^{\mathrm{T}} C \xi = \phi^2 \phi^1 = \sqrt{\mu} \psi^1 \psi^2 + \sqrt{\mu} \psi^2 \psi^1 - \phi^1 \phi^2. \tag{66}$$

The non-zero components of the multiparametric braid operator for a fourth-rank unimodular quantum matrix are

$$R = \mathrm{diag}\{1, q_{12}^{-1}, q_{13}^{-1}, q_{14}^{-1}, q_{12}, 1, q_{23}^{-1}, q_{24}^{-1}, q_{13}, q_{23}, 1, q_{34}^{-1}, q_{14}, q_{24}, q_{34}, 1\}. \tag{67}$$

From this it follows that the generators of $\mathcal{S}_{\hat{R}}$ satisfy the algebra

$$\begin{array}{ll} \phi^1 \phi^2 = q_{12} \phi^2 \phi^1 & \psi^1 \psi^2 = q_{34} \psi^2 \psi^1 \\ \phi^1 \psi^1 = q_{13} \psi^1 \phi^1 & \phi^1 \psi^2 = q_{14} \psi^2 \phi^1 \\ \phi^2 \psi^2 = q_{23} \psi^1 \phi^2 & \phi^2 \psi^2 = q_{24} \psi^2 \phi^2. \end{array} \tag{68}$$

Now the requirement of centrality of (66) relative to $\mathcal{S}_{\hat{R}}$ implies

$$q_{12} = q_{34} = 1 \qquad q_{14} = q_{23} = q_{24}^{-1} = q_{13}^{-1}. \tag{69}$$

Using (69), the commutation relations (68) are equivalent to those obtained by Chaichian and Demichev (cf the first paper cited in [10]). The invariance of (66) under the co-action map leads to

$$\begin{aligned} \delta : (\xi^{\mathrm{T}} C \xi) &= ({b^2}_i {b^1}_j - {b^1}_i {b^2}_j) \otimes \phi^i \phi^j - \sqrt{\mu} ({b^3}_i {b^4}_j - {b^4}_i {b^3}_j) \otimes \psi^i \psi^j \\ &= 1 \otimes (\xi, \xi) \end{aligned}$$

i.e.

$$\begin{array}{ll} {b^1}_1 {b^2}_2 - {b^1}_2 {b^2}_1 = 1 & {b^3}_3 {b^4}_4 - {b^3}_4 {b^4}_3 = 1 \end{array} \tag{70}$$

$$\begin{array}{ll} {b^2}_1 {b^1}_1 - {b^1}_1 {b^2}_1 = 0 & {b^2}_2 {b^1}_2 - {b^1}_2 {b^2}_2 = 0 \\ {b^3}_3 {b^4}_3 - {b^4}_3 {b^3}_3 = 0 & {b^3}_4 {b^4}_4 - {b^4}_4 {b^3}_4 = 0. \end{array} \tag{71}$$

Clearly (70) and (71) are equivalent to the condition (53), but (70) also implies automatically unimodularity of the quantum determinant of each block in the matrix B.

To determine the q_{ij} in (69) in terms of the μ-parameter in the quantum Clifford algebra, we make use of (55) to obtain

$$x^1 = \phi^2\psi^1 - \sqrt{\mu}\psi^1\phi^2 \qquad x^2 = \phi^2\psi^2 - \frac{1}{\sqrt{\mu}}\psi^2\phi^2$$
$$x'^1 = -(\phi^1\psi^2 - \sqrt{\mu}\psi^2\phi^1) \qquad x'^2 = -\mu\left(\phi^1\psi^1 - \frac{1}{\sqrt{\mu}}\psi^1\phi^1\right). \tag{72}$$

Requiring now that (72) satisfy the commutation relations (8), yields

$$q_{14} = q_{23} = q_{24}^{-1} = q_{13}^{-1} = \sqrt{\mu} \qquad q_{12} = q_{34} = 1 \tag{73}$$

and using these last relations in (49) defines completely the algebra $\mathcal{B}_R$.

We can next establish the group homomorphism by applying (56) and (59) (or, equivalently, (58)); we thus get

$$\begin{aligned}
&t^1_1 = b^2{}_2b^3{}_3 \qquad t^1{}_2 = b^2{}_2b^3{}_4 \qquad t^1_{1'} = -b^2{}_1b^3{}_4 \qquad t^1{}_{2'} = \mu^{-1}b^2{}_1b^3{}_3\\
&t^2{}_1 = b^2{}_2b^4{}_3 \qquad t^2{}_2 = b^2{}_2b^4{}_4 \qquad t^2{}_{1'} = -b^2{}_1b^4{}_4 \qquad t^2{}_{2'} = \mu^{-1}b^2{}_1b^4{}_3\\
&t^{1'}{}_1 = -b^1{}_2b^4{}_3 \qquad t^{1'}{}_2 = -b^1{}_2b^4{}_4 \qquad t^{1'}{}_{1'} = b^1{}_1b^4{}_4 \qquad t^{1'}{}_{2'} = -\mu^{-1}b^1{}_1b^4{}_3\\
&t^{2'}{}_1 = \mu b^1{}_2b^3{}_3 \qquad t^{2'}{}_2 = \mu b^1{}_2b^3{}_4 \qquad t^{2'}{}_{1'} = -\mu b^1{}_1b^3{}_4 \qquad t^{2'}{}_{2'} = b^1{}_1b^3{}_3.
\end{aligned} \tag{74}$$

It can be readily verified that (74) indeed satisfies the relations (10).

As the final step in the application of our procedure to the case $\nu = 2$, we shall derive the induced *-structure for the signatures associated with the four-dimensional Minkowski and Euclidean underlying spaces.

Minkowski space ($\nu = 2$, $h = 1$). Here, $\mu \in \mathbb{R}^+$ and

$$(x^1)^* = \psi^{1*}\phi^{2*} - \sqrt{\mu}\phi^{2*}\psi^{1*} = -b^{-1}(\phi^1\psi^2 - \sqrt{\mu}\psi^2\phi^1)$$
$$(x^2)^* = \psi^{2*}\phi^{2*} - \frac{1}{\sqrt{\mu}}\phi^{2*}\psi^{2*} = \exp(-\mathrm{i}\varphi)\left(\phi^2\psi^2 - \frac{1}{\sqrt{\mu}}\psi^2\phi^2\right)$$
$$(x'^2)^* = \mu\left(\psi^{1*}\phi^{1*} - \frac{1}{\sqrt{\mu}}\phi^{1*}\psi^{1*}\right) = \mu\exp(\mathrm{i}\varphi)\left(\phi^1\psi^1 - \frac{1}{\sqrt{\mu}}\psi^1\phi^1\right)$$

from where it follows that

$$\psi^{1*} = -\sqrt{b^{-1}}\exp(\mathrm{i}\varphi/2)\phi^1 \qquad \phi^{2*} = \sqrt{b^{-1}}\exp(-\mathrm{i}\varphi/2)\psi^2. \tag{75}$$

We now use (75) to derive the *-structure of the algebra $\mathcal{B}_R$. Recalling the comodule action requirement $(\delta(\phi^i))^* = \delta(\phi^{i*})$, and $(\delta(\psi^i))^* = \delta(\psi^{i*})$, it can be shown that

$$b^1{}_1{}^* = b^3{}_3 \qquad b^1{}_2{}^* = -b\,\mathrm{e}^{\mathrm{i}\varphi}b^3{}_4 \qquad b^2{}_1{}^* = -\frac{\mathrm{e}^{-\mathrm{i}\varphi}}{b}b^4{}_3 \qquad b^{2*}_{\ 2} = b^4{}_4. \tag{76}$$

It is a straightforward matter to verify that both the *-structures (75) and (76) are compatible with the algebras $\mathcal{S}_{\hat{R}}$ and $\mathcal{B}_R$, generated by (68) and (69) with $q_{\alpha\beta}$ given by (73).

Euclidean space ($\nu = 2$, $h = 0$). In this case μ is pure imaginary and (8) and (72) imply

$$\psi^{1*} = \sqrt{\frac{b_2}{b_1}}\psi^2 \qquad \phi^{1*} = \sqrt{\mu b_1 b_2}\phi^2 \tag{77}$$

while the comodule action requirement leads to

$$b^1{}_1{}^* = b^2{}_2 \qquad b^1{}_2{}^* = (b_1b_2)b^2{}_1 \qquad b^3{}_3{}^* = b^4{}_4 \qquad b^3{}_4{}^* = \frac{b_2}{b_1}b^4{}_3. \tag{78}$$

Consistency of this *-structure can also be checked immediately.

A.2. Quantum spin$_\mu(6-h,h)$ groups

In this case the spinor metric operator (43) becomes

$$C = H_3'H_2'H_1' - c_1H_1H_3'H_2' + c_2H_2H_3'H_1' - c_3H_3H_2'H_1' \\ -c_4H_1H_2H_3' + c_5H_1H_3H_2' - c_6H_2H_3H_1' + H_1H_2H_3 \tag{79}$$

where $c_1 = c_6 = \sqrt{\mu_{12}\mu_{13}}$, $c_2 = c_5 = \sqrt{\mu_{12}\mu_{23}}$, $c_3 = c_4 = \sqrt{\mu_{13}\mu_{23}}$. Using this, the fundamental spinor bilinear has the form

$$\xi^{\mathrm{T}}C\xi = \phi^1\psi^4 + \psi^4\phi^1 - c_1\psi^1\phi^4 - c_6\phi^4\psi^1 + c_2\psi^2\psi^3 \\ +c_5\phi^3\psi^2 - c_3\phi^3\psi^2 - c_3\psi^3\phi^2 - c_4\phi^2\psi^3. \tag{80}$$

The commutation relations between the spinors are similar to (68). However, in this case, semi-spinors of the same type do not commute.

The isotropic coordinates given in terms of the spinors are

$$\begin{aligned} x^1 &= \psi^4\psi^1 + c_5\phi^3\phi^2 - c_4\phi^2\phi^3 - c_1\psi^1\psi^4 \\ x^2 &= -\psi^4\psi^2 - c_6\mu_{12}^{-1}\phi^4\phi^2 + c_4\phi^2\phi^4 + c_2\mu_{12}^{-1}\psi^2\psi^4 \\ x^3 &= \psi^4\psi^3 + \mu_{13}^{-1}c_6\phi^4\phi^3 - c_5\mu_{23}^{-1}\phi^3\phi^4 - c_3\mu_{23}^{-1}\mu_{23}^{-1}\psi^3\psi^4 \\ x'^1 &= \phi^1\phi^4 - c_1\phi^4\phi^1 + c_2\psi^2\psi^3 - c_3\psi^3\psi^1 \\ x'^2 &= c_5\phi^3\phi^1 + \mu_{12}c_3\psi^3\psi^1 - c_1\psi^1\psi^3 - \mu_{12}\phi^1\phi^3 \\ x'^3 &= -c_4\phi^2\phi^1 - \mu_{13}c_2\psi^2\psi^1 + \mu_{23}c_1\psi^1\psi^2 + \mu_{12}\mu_{23}\phi^1\phi^2. \end{aligned} \tag{81}$$

The q parameters of deformation of the quantum Spin$(6-h,h)$ groups are related to the μ deformation parameters of the $SO_\mu(6-h,h)$ groups by

$$\begin{aligned} &q_{18} = q_{27} = q_{36} = q_{45} = 1 && q_{12} = q_{28} = q_{17}^{-1} = q_{78}^{-1} = q_{23}q_{24} \\ &q_{13} = q_{38} = q_{16}^{-1} = q_{68}^{-1} = \frac{q_{34}}{q_{23}} && q_{34} = q_{46} = q_{35}^{-1} = q_{56}^{-1} = \sqrt{\frac{\mu_{23}}{\mu_{12}}} \\ &q_{24} = q_{47} = q_{25}^{-1} = q_{57}^{-1} = \frac{1}{\sqrt{\mu_{12}\mu_{23}}} && q_{23} = q_{37} = q_{26}^{-1} = q_{67}^{-1} = \sqrt{\frac{\mu_{12}}{\mu_{23}}} \\ & && q_{14} = q_{48} = q_{15}^{-1} = q_{58}^{-1} = \frac{1}{q_{34}q_{24}}. \end{aligned} \tag{82}$$

The group homomorphism between the Spin$_\mu(6-h,h)$ groups and the $SO_\mu(6-h,h)$ groups is given by

$$\begin{aligned} t^{1'}{}_{1'} &= b^1{}_1b^4{}_4 - q_{14}b^4{}_1b^1{}_4 \\ t^{1'}{}_{2'} &= c_5^{-1}(b^1{}_3b^4{}_1 - q_{14}b^4{}_3b^1{}_1) \\ t^{1'}{}_{3'} &= \mu_{13}^{-1}\mu_{23}^{-1}(b^1{}_1b^4{}_2 - q_{14}b^4{}_1b^1{}_2) \\ t^{1'}{}_1 &= c_5^{-1}(b^1{}_3b^4{}_2 - q_{14}b^4{}_3b^1{}_2) \\ t^{1'}{}_2 &= \mu_{12}c_6^{-1}(b^1{}_4b^4{}_2 - q_{14}b^4{}_4b^1{}_2) \\ t^{1'}{}_3 &= \mu_{13}c_6^{-1}(b^1{}_4b^4{}_3 - q_{14}b^4{}_4b^1{}_3) \\ t^{2'}{}_{1'} &= c_5b^3{}_1b^1{}_4 - \mu_{12}b^1{}_1b^3{}_4 \\ t^{2'}{}_{2'} &= -c_5^{-1}q_{13}(c_5b^3{}_1b^1{}_3 - \mu_{12}b^1{}_1b^3{}_3) \\ t^{2'}{}_{3'} &= \mu_{13}^{-1}\mu_{23}^{-1}(c_5b^3{}_1b^1{}_2 - \mu_{12}b^1{}_1b^3{}_2) \\ t^{2'}{}_1 &= -c_5^{-1}q_{23}(c_5b^3{}_2b^1{}_3 - \mu_{12}b^1{}_2b^3{}_3) \\ t^{2'}{}_2 &= -c_6^{-1}q_{24}(c_5b^3{}_2b^1{}_4 - \mu_{12}b^1{}_2b^3{}_4) \end{aligned}$$

$$
\begin{aligned}
&t^{2'}{}_3 = -c_6^{-1} q_{34}\mu_{13}(c_5 b^3{}_1 b^1{}_4 - \mu_{12} b^1{}_1 b^3{}_4)\\
&t^{3'}{}_{1'} = \mu_{13}\mu_{23} b^1{}_1 b^2{}_4 - c_4 b^2{}_1 b^1{}_4\\
&t^{3'}{}_{2'} = -q_{13} c_5^{-1}(\mu_{13}\mu_{23} b^1{}_1 b^2{}_3 - c_4 b^2{}_1 b^1{}_3)\\
&t^{3'}{}_{3'} = (b^1{}_1 b^2{}_2 - c_4 \mu_{13}^{-1}\mu_{23}^{-1} b^2{}_1 b^1{}_2)\\
&t^{3'}{}_1 = -q_{23} c_5^{-1}(\mu_{13}\mu_{23} b^1{}_2 b^2{}_3 - c_4 b^2{}_2 b^1{}_3)\\
&t^{3'}{}_2 = -q_{24}\mu_{12} c_6^{-1}(\mu_{13}\mu_{23} b^1{}_2 b^2{}_4 - c_4 b^2{}_2 b^1{}_4)\\
&t^{3'}{}_3 = -q_{34}\mu_{13} c_6^{-1}(\mu_{13}\mu_{23} b^1{}_3 b^2{}_4 - c_4 b^2{}_3 b^1{}_4)\\
&t^{1}{}_{1'} = c_5 b^3{}_1 b^2{}_4 - c_4 b^2{}_1 b^3{}_4\\
&t^{1}{}_{2'} = -q_{13} c_5^{-1}(c_5 b^3{}_1 b^2{}_3 - c_4 b^2{}_1 b^3{}_3)\\
&t^{1}{}_{3'} = \mu_{13}^{-1}\mu_{23}^{-1}(c_5 b^3{}_1 b^2{}_2 - c_4 b^2{}_1 b^3{}_2)\\
&t^{1}{}_1 = -q_{13} c_5^{-1}(c_5 b^3{}_2 b^2{}_3 - c_4 b^2{}_2 b^3{}_3)\\
&t^{1}{}_2 = -q_{24}\mu_{12} c_6^{-1}(c_5 b^3{}_2 b^2{}_4 - c_4 b^2{}_2 b^3{}_4)\\
&t^{1}{}_3 = -q_{34}\mu_{13} c_6^{-1}(c_5 b^3{}_3 b^2{}_4 - c_4 b^2{}_3 b^3{}_4)\\
&t^{2}{}_{1'} = \mu_{12}^{-1} c_6 b^4{}_1 b^2{}_4 - c_4 b^2{}_1 b^4{}_4\\
&t^{2}{}_{2'} = -q_{13} c_5^{-1}(\mu_{12}^{-1} c_6 b^4{}_1 b^2{}_3 - c_4 b^2{}_1 b^4{}_3)\\
&t^{2}{}_{3'} = \mu_{13}^{-1}\mu_{23}^{-1}(\mu_{12}^{-1} c_6 b^4{}_1 b^2{}_2 - c_4 b^2{}_1 b^4{}_2)\\
&t^{2}{}_1 = -q_{23} c_5^{-1}(\mu_{12}^{-1} c_6 b^4{}_2 b^2{}_3 - c_4 b^2{}_2 b^4{}_3)\\
&t^{2}{}_2 = -q_{24}\mu_{12} c_6^{-1}(\mu_{12}^{-1} c_6 b^4{}_2 b^2{}_4 - c_4 b^2{}_2 b^4{}_4)\\
&t^{2}{}_3 = -q_{34}\mu_{13} c_6^{-1}(\mu_{12}^{-1} c_6 b^4{}_3 b^2{}_4 - c_4 b^2{}_3 b^4{}_4)\\
&t^{3}{}_{1'} = \mu_{13}^{-1} c_6 b^4{}_1 b^3{}_4 - \mu_{23}^{-1} c_5 b^3{}_1 b^4{}_4\\
&t^{3}{}_{2'} = -q_{13} c_5^{-1}(\mu_{13}^{-1} c_6 b^4{}_1 b^3{}_3 - \mu_{23}^{-1} c_5 b^3{}_1 b^4{}_3)\\
&t^{3}{}_{3'} = \mu_{13}^{-1}\mu_{23}^{-1}(\mu_{13}^{-1} c_6 b^4{}_1 b^3{}_2 - \mu_{23}^{-1} c_5 b^3{}_1 b^4{}_2)\\
&t^{3}{}_1 = -q_{23} c_5^{-1}(\mu_{13}^{-1} c_6 b^4{}_2 b^3{}_3 - \mu_{23}^{-1} c_5 b^3{}_2 b^4{}_3)\\
&t^{3}{}_2 = -q_{24}\mu_{12} c_6^{-1}(\mu_{13}^{-1} c_6 b^4{}_2 b^3{}_4 - \mu_{23}^{-1} c_5 b^3{}_2 b^4{}_4)\\
&t^{3}{}_3 = -q_{34}\mu_{13} c_6^{-1}(\mu_{13}^{-1} c_6 b^4{}_3 b^3{}_4 - \mu_{23}^{-1} c_5 b^3{}_3 b^4{}_4).
\end{aligned}
$$

The *-structure for the quantum spinors associated to a six-dimensional Minkowski space is

$$
\begin{aligned}
&\phi^{1*} = \sqrt{\mu_{23}}\,\phi^2 \qquad \phi^{4*} = \sqrt{\mu_{12}}\,\phi^3\\
&\psi^{1*} = (\mu_{12}^2\mu_{23}/\mu_{13})^{1/4}\psi^2 \qquad \psi^{4*} = (\mu_{13}\mu_{23})^{1/4}\psi^3.
\end{aligned}
\tag{83}
$$

A.3. Quantum Spin$_\mu(8-h,h)$ groups

Applying requirements (i)–(iv) of our general procedure to the case $\nu = 4$, we obtain

$$
\begin{aligned}
C = {} & H_1H_2H_3H_4 - a_1H_1H_4'H_3'H_2' + a_2H_2H_4'H_3'H_1' - a_3H_3H_4'H_2'H_1' + a_4H_4H_3'H_2'H_1'\\
& - a_5H_1H_2H_4'H_3' + a_6H_1H_3H_4'H_2' - a_7H_1H_4H_3'H_2' - a_8H_2H_3H_4'H_1'\\
& + a_9H_2H_4H_3'H_1' - a_{10}H_3H_4H_2'H_1' + a_{11}H_1H_2H_3H_4' - a_{12}H_1H_2H_4H_3'\\
& - a_{13}H_2H_3H_4H_1' + a_{14}H_1H_3H_4H_2' + H_4'H_3'H_2'H_1'
\end{aligned}
$$

with

$$
a_1 = a_{13} = \sqrt{\mu_{14}\mu_{13}\mu_{12}} \qquad a_2 = a_{14} = \sqrt{\mu_{24}\mu_{23}\mu_{12}}
$$

$$a_3 = a_{12} = \sqrt{\mu_{13}\mu_{23}\mu_{34}} \qquad a_4 = a_{11} = \sqrt{\mu_{14}\mu_{24}\mu_{34}}$$
$$a_5 = a_{10} = \sqrt{\mu_{14}\mu_{13}\mu_{24}\mu_{23}} \qquad a_6 = a_9 = \sqrt{\mu_{14}\mu_{23}\mu_{12}\mu_{34}}$$
$$a_7 = a_8 = \sqrt{\mu_{24}\mu_{34}\mu_{13}\mu_{12}}. \tag{84}$$

The fundamental spinor bilinear is given by

$$\xi^{\mathrm{T}}C\xi = \phi^1\phi^8 + \phi^8\phi^1 - a_5\phi^2\phi^7 - a_{10}\phi^7\phi^2 + a_6\phi^3\phi^6 + a_9\phi^6\phi^3 - a_7\phi^4\phi^5 - a_8\phi^5\phi^4$$
$$-a_1\psi^1\psi^8 - a_{13}\psi^8\psi^1 + a_2\psi^2\psi^7 + a_{14}\psi^7\psi^2 - a_3\psi^3\psi^6 - a_{12}\psi^6\psi^3$$
$$+a_4\psi^4\psi^5 + a_{11}\psi^5\psi^4. \tag{85}$$

The isotropic coordinates expressed in terms of spinors are

$$x^1 = \phi^8\psi^1 - a_1\psi^1\phi^8 + a_{14}\psi^7\phi^2 - a_5\phi^2\psi^7 + a_6\phi^3\psi^6 - a_{12}\psi^6\phi^3$$
$$+a_{11}\psi^5\phi^4 - a_7\phi^4\psi^5 \tag{86}$$
$$x^2 = \phi^8\psi^2 - a_2\mu_{12}^{-1}\psi^2\phi^8 + a_{13}\mu_{12}^{-1}\psi^8\phi^2 - a_5\phi^2\psi^8 + a_8\mu_{12}^{-1}\phi^5\psi^6$$
$$-a_{12}\psi^6\phi^5 + a_{11}\psi^5\phi^6 - a_9\mu_{12}^{-1}\phi^4\psi^5 \tag{87}$$
$$x^3 = \phi^8\psi^3 - a_3\mu_{13}^{-1}\mu_{23}^{-1}\psi^3\phi^8 + a_{13}\mu_{13}^{-1}\psi^8\phi^3 - a_6\mu_{23}^{-1}\phi^3\psi^8 + a_8\mu_{13}^{-1}\phi^5\psi^7$$
$$-a_{14}\mu_{23}^{-1}\psi^7\phi^5 + a_{11}\psi^5\phi^7 - a_{10}\mu_{13}^{-1}\mu_{23}^{-1}\phi^7\psi^5 \tag{88}$$
$$x^4 = \phi^8\psi^4 - a_4\mu_{14}^{-1}\mu_{24}^{-1}\mu_{34}^{-1}\psi^4\phi^8 + a_{13}\mu_{14}^{-1}\psi^8\phi^4 - a_7\mu_{24}^{-1}\mu_{34}^{-1}\phi^4\psi^8 + a_9\mu_{14}^{-1}\mu_{34}^{-1}\phi^6\psi^7$$
$$-a_{14}\mu_{24}^{-1}\psi^7\phi^6 + a_{12}\mu_{34}^{-1}\psi^6\phi^7 - a_{10}\mu_{14}^{-1}\mu_{24}^{-1}\phi^7\psi^6 \tag{89}$$
$$x'^1 = a_2\psi^2\phi^7 - a_{10}\phi^7\psi^2 + a_3\psi^3\phi^6 + a_9\phi^6\psi^3 + a_4\psi^4\psi^5 - a_8\phi^5\psi^4$$
$$+a_{13}\psi^8\phi^1 + \phi^1\psi^8 \tag{90}$$
$$x'^2 = a_{14}\psi^7\phi^1 - \mu_{12}\phi^1\psi^7 + a_1\psi^1\phi^7 + a_{10}\mu_{12}\phi^7\psi^1 + a_3\mu_{12}\psi^3\phi^4 - a_7\phi^4\psi^3$$
$$-a_4\mu_{12}\psi^4\phi^3 + a_6\phi^3\psi^4 \tag{91}$$
$$x'^3 = -a_{12}\psi^6\phi^1 + \mu_{13}\mu_{23}\phi^1\psi^6 + a_1\mu_{23}\psi^1\phi^6 - a_9\mu_{13}\phi^6\psi^1 - a_2\mu_{13}\psi^2\psi^4$$
$$+a_2\mu_{23}\phi^4\psi^2 + a_4\mu_{13}\mu_{23}\psi^4\phi^2 - a_5\phi^2\psi^4 \tag{92}$$
$$x'^4 = a_{11}\psi^5\phi^1 - \mu_{14}\mu_{24}\mu_{34}\phi^1\psi^5 - a_1\mu_{24}\mu_{34}\psi^1\phi^5 + a_8\mu_{14}\phi^5\psi^1 + a_2\mu_{14}\mu_{24}\psi^2\phi^3$$
$$-a_6\mu_{24}\phi^3\psi^2 - a_3\mu_{14}\mu_{24}\psi^3\phi^2 + a_5\mu_{34}\phi^2\psi^3. \tag{93}$$

The relationship between the deformation parameters of the $\mathrm{Spin}_\mu(8-h,h)$ groups and the μ-parameters of the $SO_\mu(8-h,h)$ groups is

$$q_{18} = q_{27} = q_{36} = q_{45} = q_{9,16} = q_{10,15} = q_{11,14} = q_{12,13} = 1 \tag{94}$$
$$q_{12} = q_{28} = q_{17}^{-1} = q_{78}^{-1} = \sqrt{\frac{1}{\mu_{13}\mu_{14}\mu_{23}\mu_{24}}}$$
$$q_{13} = q_{38} = q_{16}^{-1} = q_{68}^{-1} = \sqrt{\frac{\mu_{23}}{\mu_{12}\mu_{14}\mu_{34}}}$$
$$q_{14} = q_{48} = q_{15}^{-1} = q_{58}^{-1} = \sqrt{\frac{\mu_{24}\mu_{34}}{\mu_{12}\mu_{13}}}$$
$$q_{23} = q_{37} = q_{26}^{-1} = q_{67}^{-1} = \sqrt{\frac{\mu_{12}\mu_{24}}{\mu_{13}\mu_{34}}}$$
$$q_{24} = q_{47} = q_{25}^{-1} = q_{57}^{-1} = \sqrt{\frac{\mu_{12}\mu_{23}\mu_{34}}{\mu_{14}}}$$
$$q_{34} = q_{46} = q_{35}^{-1} = q_{56}^{-1} = \sqrt{\frac{\mu_{13}\mu_{24}}{\mu_{14}\mu_{23}}}$$

$$q_{9,10} = q_{10,16} = q_{9,15}^{-1} = q_{15,16}^{-1} = \sqrt{\frac{\mu_{13}\mu_{14}}{\mu_{23}\mu_{24}}}$$
$$q_{9,11} = q_{11,16} = q_{9,14}^{-1} = q_{14,16}^{-1} = \sqrt{\frac{\mu_{12}\mu_{23}\mu_{14}}{\mu_{34}}}$$
$$q_{9,12} = q_{12,16} = q_{9,13}^{-1} = q_{13,16}^{-1} = \sqrt{\mu_{12}\mu_{24}\mu_{13}\mu_{34}}$$
$$q_{10,11} = q_{11,15} = q_{10,14}^{-1} = q_{14,15}^{-1} = \sqrt{\frac{\mu_{13}\mu_{24}}{\mu_{12}\mu_{34}}}$$
$$q_{10,12} = q_{12,15} = q_{10,13}^{-1} = q_{13,15}^{-1} = \sqrt{\frac{\mu_{14}\mu_{23}\mu_{34}}{\mu_{12}}}$$
$$q_{11,12} = q_{12,14} = q_{11,13}^{-1} = q_{13,14}^{-1} = \sqrt{\frac{\mu_{14}\mu_{24}}{\mu_{13}\mu_{23}}}$$
$$q_{1,9} = q_{8,16} = q_{1,16}^{-1} = q_{8,9}^{-1} = \sqrt{\frac{1}{\mu_{14}\mu_{13}\mu_{12}}}$$
$$q_{1,10} = q_{8,15} = q_{1,15}^{-1} = q_{8,10}^{-1} = \sqrt{\frac{\mu_{12}}{\mu_{23}\mu_{24}}}$$
$$q_{1,11} = q_{8,14} = q_{1,14}^{-1} = q_{8,11}^{-1} = \sqrt{\frac{\mu_{13}\mu_{23}}{\mu_{34}}}$$
$$q_{1,12} = q_{8,13} = q_{1,13}^{-1} = q_{8,12}^{-1} = \sqrt{\mu_{14}\mu_{24}\mu_{34}}$$
$$q_{2,9} = q_{7,16} = q_{2,16}^{-1} = q_{7,9}^{-1} = \sqrt{\mu_{12}\mu_{24}\mu_{23}}$$
$$q_{2,10} = q_{7,15} = q_{2,15}^{-1} = q_{7,10}^{-1} = \sqrt{\frac{\mu_{13}\mu_{14}}{\mu_{12}}}$$
$$q_{2,11} = q_{7,14} = q_{2,14}^{-1} = q_{7,11}^{-1} = \sqrt{\frac{\mu_{14}\mu_{24}}{\mu_{34}}}$$
$$q_{2,12} = q_{7,13} = q_{2,13}^{-1} = q_{7,12}^{-1} = \sqrt{\mu_{13}\mu_{23}\mu_{34}}$$
$$q_{3,9} = q_{6,9} = q_{3,16}^{-1} = q_{6,16}^{-1} = \sqrt{\frac{\mu_{13}\mu_{34}}{\mu_{24}}}$$
$$q_{3,10} = q_{6,15} = q_{3,15}^{-1} = q_{6,10}^{-1} = \sqrt{\frac{\mu_{14}\mu_{34}}{\mu_{24}}}$$
$$q_{3,11} = q_{6,14} = q_{3,14}^{-1} = q_{6,11}^{-1} = \sqrt{\frac{\mu_{12}\mu_{14}}{\mu_{13}}}$$
$$q_{3,12} = q_{6,13} = q_{3,13}^{-1} = q_{6,12}^{-1} = \sqrt{\frac{\mu_{24}\mu_{12}}{\mu_{23}}}$$
$$q_{4,9} = q_{5,16} = q_{4,16}^{-1} = q_{5,9}^{-1} = \sqrt{\frac{\mu_{14}}{\mu_{34}\mu_{24}}}$$
$$q_{4,10} = q_{5,15} = q_{4,15}^{-1} = q_{5,10}^{-1} = \sqrt{\frac{\mu_{13}}{\mu_{23}\mu_{34}}}$$
$$q_{4,11} = q_{5,14} = q_{4,14}^{-1} = q_{5,11}^{-1} = \sqrt{\frac{\mu_{23}\mu_{12}}{\mu_{24}}}$$
$$q_{4,12} = q_{5,13} = q_{4,13}^{-1} = q_{5,12}^{-1} = \sqrt{\frac{\mu_{12}\mu_{13}}{\mu_{14}}}.$$

The *-structure for the quantum spinors in an eight-dimensional space with Minkowski signature is

$$\begin{array}{llll} (\phi^1)^* = \alpha_1\psi_5 & (\phi^2)^* = \alpha_2\psi_3 & (\phi^3)^* = \alpha_3\psi_2 & (\phi^4)^* = \alpha_4\psi_8 \\ (\phi^5)^* = \alpha_4\psi_1 & (\phi^6)^* = \alpha_3\psi_7 & (\phi^7)^* = \alpha_2\psi_6 & (\phi^8)^* = \alpha_4\psi_8 \end{array} \tag{95}$$

with

$$\begin{array}{ll} \alpha_1 = (\mu_{14}\mu_{24}\mu_{34})^{1/4} & \alpha_2 = -\left(\dfrac{\mu_{13}^2\mu_{23}^2\mu_{34}}{\mu_{24}\mu_{14}}\right)^{1/4} \\ \alpha_3 = \left(\dfrac{\mu_{12}^2\mu_{23}^2\mu_{24}}{\mu_{24}\mu_{14}}\right)^{1/4} & \alpha_4 = -\left(\dfrac{\mu_{12}^2\mu_{13}^2\mu_{14}}{\mu_{24}\mu_{34}}\right)^{1/4}. \end{array} \tag{96}$$

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